Separation Solutions for the Similar Laminar Boundary Layer

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The complete spectrum of similar laminar boundary-layer solutions exhibiting separation, pressure gradient, and mass transfer (blowing or suction) have been obtained. The resulting map includes variation of pressure gradient parameter and heat-transfer parameter as functions of mass transfer rate and wall enthalpy ratio. The solutions range from the limiting Blasius blow-off to an identification of asymptotic suction solutions. These latter solutions are obtained analytically.

Nomenclature

a,b	= curve fit constants
c	= constant of integration
f	= transformed stream function
g	= stagnation enthalpy ratio, h_s/h_{se}
h	= static enthalpy
q_w	= wall heat-transfer rate
r	= radius
s	= transformed streamwise coordinate
u	= streamwise velocity
v	= normal velocity
x	= streamwise coordinate
y	= normal coordinate
$\hat{\beta}$	= pressure gradient parameter
δ^*	= displacement thickness
η	= transformed normal coordinate
θ	= momentum thickness
θ_H	= energy thickness
κ	= characteristic of asymptotic solution
μ	= viscosity
ρ	= density

Subscripts

e	= conditions external to the boundary layer
$k,k+1$	= iteration indices
s	= stagnation conditions
tr	= transformed quantity
w	= conditions at the wall

Superscripts

j	= 0 or 1 for two-dimensional or axisymmetric flow respectively
$(\ )'$	= differentiation with respect to η

1. Introduction

IN recent reviews of the laminar boundary-layer literature[1,2] a substantial incompleteness for solutions with pressure gradients and the combined effects of mass transfer and separation was noted. Indeed, except for the limited results of Cohen and Reshotko[3] and Emmons and Leigh,[4] corresponding for the former to separation with zero mass transfer and for the latter to separation at zero pressure gradient, a complete spectrum of similar solutions does not exist.

The present study was undertaken in an effort to fill partially this gap. In general, solutions have been sought to the boundary-layer equations described in Ref. 3. For a prescribed wall enthalpy ratio, the pair of values for the blowing or suction rate and pressure gradient parameter was determined so that the resulting solution exhibited blowoff or zero wall shear. To keep the number of parameters within reason we follow common practice and employ simplified transport phenomena, i.e. the mass-density-viscosity product was taken constant, and for the blowing cases the injectant properties were assumed to be those of the external flow.

An iterative method of solution, first suggested by Weyl[5] and later used by many other authors (see e.g. Ref. 6), was used here. While convergence could not be proved the solutions so obtained were found to satisfy reasonable tolerances, even for very crude initial distributions, within a limited number of iterates.

The range of parameters selected provides a reasonably complete map of the eigenvalues for the blow-off solutions. We make the observation here that for the case of blowing a definitive upper limit on the rate is offered in Ref. 4 for a uniform external flow. No such endpoint exists for suction; thus we have arbitrarily selected a convenient numerical upper limit as suggested by other typical studies. However, as will be observed later, even for this minimum suction solution, an asymptotic solution for the energy equation, related to those presented in Refs. 7 and 8, appears to exist and is developed analytically. The corresponding minimum solution for the momentum equation has recently been attempted by Kassoy and Libby.[9] Consequently, within reasonable tolerances on accuracy the entire spectrum of separation solutions may be considered determined.

Studies related to this but of a more applied nature may also be cited. Here we recall the fact that many of the Cohen-Reshotko[3] lower branch and separated solutions have been used successfully in inviscid-viscous interaction analyses and consequently the results presented here may be of some value in this regard. Typical research in this area may be found in Refs. 10, 11, and 12.

2. Analysis and Solution Technique

In this section the basic describing equations and the solution technique will be presented. Included in the discussion will be the pertinent physical quantities of interest.

Basic Equations

Application of the Levy-Lees transformation[13] to the classical compressible boundary-layer equations with the further assumptions of local flow similarity and simplified transport phenomena results in the familiar relations from Ref. 3.

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0 \tag{1}$$

$$g'' + fg' = 0 \tag{2}$$

Received March 17, 1969; revision received October 8, 1969. The research presented here was partially supported by the Aerospace Research Laboratories, Office of Aerospace Research, USAF, Wright-Patterson Air Force Base, Ohio, under Contract F33615-68-C-1184. The authors are indebted to H. Joslyn who participated in the early stages of the work and to G. D. Bleich for his assistance with the numerical computations.

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$$\hat{\beta} = (2s/u_e)(du_e/ds)(1 - u_e^2/2h_{se})^{-1} \quad (3)$$

where primes denote differentiation with respect to the similarity variable

$$\eta = [\rho_e u_e r^j/(2s)^{1/2}] \int_0^y (\rho/\rho_e)dy \quad (4a)$$

where

$$s = \int_0^x \rho_e \mu_e u_e r^{2j} dx \quad (4b)$$

and where $j = 0$ or 1 for two-dimensional or axisymmetric flows respectively.

Consider next the applicable boundary conditions for the separation problem. At the wall, $\eta = 0$

$$f'(0) = 0$$
$$f''(0) = 0 \quad (5)$$
$$g(0) = g_w$$

At the edge of the boundary layer, as $\eta \to \infty$

$$\lim_{\eta \to \infty} f'(\eta) \to 1 \quad (6)$$
$$\lim_{\eta \to \infty} g(\eta) \to 1$$

Note that the second of Eq. (5) is equivalent to prescribing zero shear.

For given wall temperature, g_w, and mass transfer rate, f_w, we seek the pair of values $\hat{\beta}$, the pressure gradient parameter, and $g'(0)$, the heat-transfer rate such that the set of Eqs. (1, 2, 5, and 6) are satisfied.

The pertinent physical quantities may be summarized as follows:

Transformed momentum thickness:

$$\theta_{tr} = [\rho_e u_e r^j/(2s)^{1/2}]\theta = \int_0^\infty f'(1 - f')d\eta \quad (7)$$

Transformed energy thickness:

$$\theta_{Htr} = [\rho_e u_e r^j/(2s)^{1/2}]\theta_H = \int_0^\infty f'(1 - g)d\eta \quad (8)$$

Transformed displacement thickness:

$$\delta_{tr}^* = [\rho_e u_e r^j/(2s)^{1/2}](1 - u_e^2/2h_{se})\delta^* = \int_0^\infty (g - f')d\eta + (u_e^2/2h_{se})\theta_{tr} \quad (9)$$

For convenience we introduce

$$\delta_{tr0}^* = \int_0^\infty (g - f')d\eta \quad (10)$$

and only tabulate δ_{tr0}^* as part of the final results.

Blowing or suction rate:

$$(\rho_w v_w/\rho_e u_e)\ [(2s)^{1/2}/\mu_e r^{2j}] = -f_w \quad (11)$$

Heat-transfer rate:

$$q_w[(2s)^{1/2}/\rho_e \mu_e u_e r^j h_{se}] = g'(0) \quad (12)$$

Characterization of asymptotic solution as $\eta \to \infty$:

As $\eta \to \infty$ the stream function, f, behaves as:

$$f \to \eta - \kappa$$

It is sometimes useful in application studies to have specific values for κ. These have also been obtained.

We now proceed to the solution of Eqs. (1) and (2); we treat them formally as equations for f'' and g', all other terms presumed known. Then by integration and making use of the boundary conditions we have for the momentum equation:

$$f'' = \hat{\beta}\left[\exp\left(-\int_0^\eta f d\eta\right)\right]\left[\int_0^\eta (f'^2 - g)\times \exp\left(\int_0^\eta f d\eta\right)d\eta\right] \quad (13)$$

$$f' = \hat{\beta}\int_0^\eta \exp\left(-\int_0^\eta f d\eta\right)\left[\int_0^\eta (f'^2 - g)\times \exp\left(\int_0^\eta f d\eta\right)d\eta\right]d\eta \quad (14)$$

and

$$f = \int_0^\eta f'd\eta + f_w \quad (15)$$

where the constants of integration in Eqs. (13) and (14) have been set equal to zero because of the first two conditions in Eq. (5). The edge condition on f' in Eq. (6) may be used for evaluation of $\hat{\beta}$, viz:

$$\hat{\beta} = \left\{\int_0^\infty \exp\left(-\int_0^\eta f d\eta\right)\left[\int_0^\eta (f'^2 - g)\times \exp\left(\int_0^\eta f d\eta\right)d\eta\right]d\eta\right\}^{-1} \quad (16)$$

Similarly, for the energy equation, we may write

$$g' = c\exp\left(-\int_0^\eta f d\eta\right) \quad (17)$$

and

$$g = c\int_0^\eta \left[\exp\left(-\int_0^\eta f d\eta\right)\right]d\eta + g_w \quad (18)$$

where the constant c may be evaluated by making use of the second of Eqs. (6):

$$c = (1 - g_w)\left\{\int_0^\infty\left[\exp\left(-\int f d\eta\right)\right]d\eta\right\}^{-1} \quad (19)$$

Provided then, that we have correct distributions for the dependent variables, these integral equations are satisfied identically. This suggests an iteration procedure, which we indicate by subscripts k and $k + 1$ for successive approximations; the equations are then conveniently ordered by:

$$f_k = \int_0^\eta f'_k d\eta + f_w \quad (20)$$

$$g_k = (1 - g_w)\left\{\int_0^\eta \left[\exp\left(-\int_0^\eta f_k d\eta\right)\right]d\eta\right\}\times \left\{\int_0^\infty\left[\exp\left(-\int f_k d\eta\right)\right]d\eta\right\}^{-1} + g_w \quad (21)$$

$$f'_{k+1} = \int_0^\eta \exp\left(-\int_0^\eta f_k d\eta\right)\left[\int_0^\eta (f'^2_k - g_k)\times \exp\left(\int_0^\eta f_k d\eta\right)d\eta \div \int_0^\infty \exp\left(-\int_0^\eta f_k d\eta\right)\times \left[\int_0^\eta (f'^2_k - g_k)\exp\left(\int_0^\eta f_k d\eta\right)d\eta\right]d\eta \quad (22)$$

Equivalent relations can be generated for the distributions of $f''(\eta)$ and $g'(\eta)$.

The iteration proceeds as follows: An initial distribution is assumed for f', call it $f'_{(0)}$. Then from Eqs. (20) and (21) with given values of f_w and g_w the corresponding distribution for $f_{(0)}$ and $g_{(0)}$ can be determined. These, when inserted into Eq. (22), yield the next iterate for $f' \equiv f'_{(1)}$. The procedure can continue until successive iterates agree to within a prescribed tolerance. For the present application a case was considered concluded when:

$$|f'_{k+1}(\eta) - f'_k(\eta)|/f'_k(\eta) \leq 10^{-4} \quad (23)$$

Note that the value of $\hat{\beta}$ is obtained during the solution since the integral in Eq. (16) is required for the complete determination of the velocity distribution f'. The integration

Table 1 Eigenvalues for blowing solutions

g_w	$-f_w$	$-\hat{\beta}$	g' (0)	$\delta_{tro}{}^*$	θ_{tr}	$\theta_{H tr}$	κ
0.0	0.0	0.32718	0.24813	1.3176	0.63939	0.24813	3.4551
	0.2	0.25708	0.13256	1.1982	0.68381	0.33256	4.2224
	0.4	0.19531	0.05036	0.98383	0.73587	0.45036	5.3146
	0.6	0.1368	0.00806	0.64481	0.79725	0.60806	7.2807
	0.8	0.0597	0.00000	0.18765	0.86265	0.80000	15.827
0.001	0.0	0.32670	0.24800	1.3182	0.63969	0.24800	3.4512
	0.2	0.25710	0.13276	1.1985	0.68394	0.33256	4.2149
	0.4	0.19530	0.05075	0.98395	0.73590	0.45035	5.2950
	0.6	0.13670	0.0086	0.64497	0.79721	0.60803	7.1666
	0.8	0.05750	0.00018	0.19014	0.86040	0.79938	10.818
0.010	0.0	0.32660	0.24792	1.3185	0.63963	0.24792	3.4160
	0.2	0.25690	0.13440	1.1992	0.68381	0.33240	4.1507
	0.4	0.19490	0.05393	0.98587	0.73555	0.44993	5.1396
	0.6	0.13480	0.01230	0.65393	0.79539	0.60630	6.5630
	0.8	0.04970	0.00099	0.23838	0.85429	0.79299	8.6497
0.1	0.0	0.3211	0.24155	1.34401	0.63582	0.24155	3.1364
	0.2	0.2481	0.14110	1.2471	0.67741	0.32110	3.7140
	0.4	0.1790	0.06819	1.0856	0.72385	0.42819	4.3913
	0.6	0.1062	0.02406	0.87426	0.77515	0.56406	5.2568
	0.8	0.0246	0.00312	0.72862	0.83849	0.72312	7.0397
0.2	0.0	0.30888	0.22618	1.4059	0.62833	0.22618	2.9267
	0.1	0.26965	0.17899	1.3827	0.64741	0.25899	3.1768
	0.2	0.23169	0.13697	1.3504	0.66754	0.29697	3.4448
	0.4	0.15781	0.06999	1.2625	0.71151	0.38999	4.0590
	0.6	0.08412	0.02622	1.1761	0.76313	0.50622	4.9000
	0.8	0.01606	0.00340	1.2777	0.83391	0.64340	6.7734
	0.85	0.00388	0.00074	1.5239	0.85923	0.68074	8.3344
0.4	0.0	0.27805	0.18087	1.5931	0.61358	0.18087	2.6693
	0.1	0.23710	0.14508	1.6111	0.63179	0.20508	2.9041
	0.2	0.19783	0.11276	1.6301	0.65133	0.23276	3.1587
	0.4	0.12424	0.05960	1.6846	0.69574	0.29960	3.7625
	0.6	0.05892	0.02287	1.8196	0.75149	0.38287	4.6321
	0.8	0.00953	0.00289	2.3787	0.83058	0.48289	6.6046
	0.85	0.00216	0.00061	3.0268	0.85837	0.51061	8.2198
0.6	0.0	0.24777	0.12518	1.8269	0.60173	0.12518	2.5203
	0.1	0.20754	0.10096	1.8862	0.62017	0.14096	2.7541
	0.2	0.16962	0.07891	2.0151	0.65247	0.87891	3.0101
	0.4	0.10134	0.04214	2.1420	0.68666	0.20214	3.6246
	0.6	0.04520	0.01624	2.4805	0.74583	0.25624	4.5196
	0.8	0.00678	0.00202	3.4807	0.82924	0.32202	6.5404
	0.85	0.00150	0.00042	4.4092	0.85789	0.34042	8.1781
0.8	0.0	0.22134	0.06412	2.0855	0.59282	0.06412	2.4246
	0.1	0.18299	0.05185	2.1850	0.61180	0.07185	2.6605
	0.2	0.14742	0.04064	2.3014	0.63253	0.08064	2.9198
	0.4	0.08524	0.02180	2.6139	0.68083	0.10180	3.5450
	0.6	0.03662	0.00841	3.1480	0.74246	0.12841	4.4577
	0.8	0.00527	0.00104	4.5831	0.82853	0.16104	6.5066
	0.85	0.00115	0.00022	5.8512	0.85771	0.17022	8.1565
1.0	0.0	0.19901	0.0	2.3588	0.58606	0.0	2.3588
	0.2	0.12992	0.0	2.6595	0.62698	0.0	2.8594
	0.4	0.07344	0.0	3.0933	0.67688	0.0	3.4932
	0.6	0.03077	0.0	3.8183	0.74027	0.0	4.4182
	0.8	0.00430	0.0	5.6855	0.82801	0.0	6.4855
	0.85	0.00093	0.0	7.2933	0.85758	0.0	8.1433
1.5	0.0	0.15749	−0.16720	3.0758	0.57495	−0.16720	2.2598
	0.2	0.09963	−0.10667	3.5798	0.61824	−0.20667	2.7710
	0.4	0.05441	−0.0575	4.3072	0.67085	−0.25750	3.4191
	0.6	0.02198	−0.02216	5.5003	0.73707	−0.32216	4.3629
	0.8	0.00296	−0.00269	8.2399	0.82685	−0.40269	6.4568
	0.85	0.00063	−0.00055	12.098	0.85816	−0.42555	8.1254
2.0	0.0	0.12962	−0.33913	3.8163	0.56833	−0.33913	2.2053
	0.2	0.08054	−0.21672	4.5174	0.61320	−0.41672	2.7233
	0.4	0.04316	−0.11694	5.531	0.66755	−0.51694	3.3796
	0.6	0.01709	−0.04505	6.4192	0.72206	−0.64505	4.3340
	0.8	0.00225	−0.00545	11.199	0.82712	−0.80545	6.4422
	0.85	0.00048	−0.00111	14.503	0.85737	−0.85111	8.1163

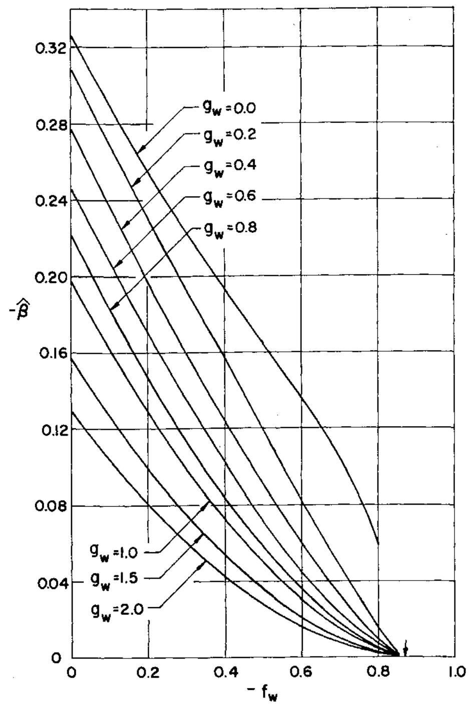

Fig. 1a Variation of pressure gradient parameter with blowing rate.

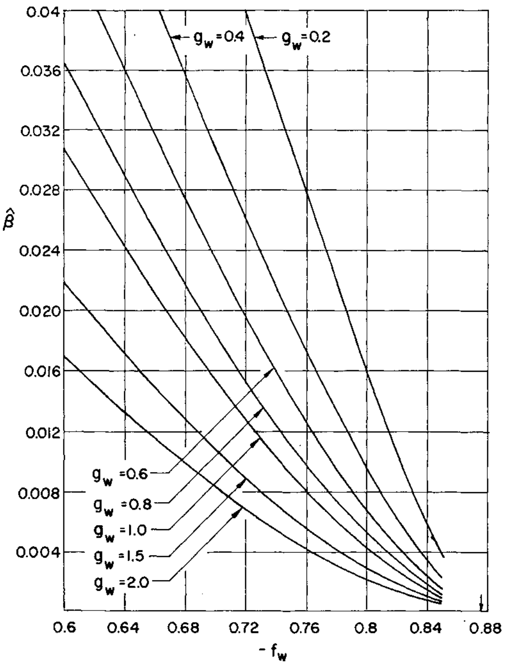

Fig. 1b Variation of pressure gradient parameter for $-f_w \to 0.87574$.

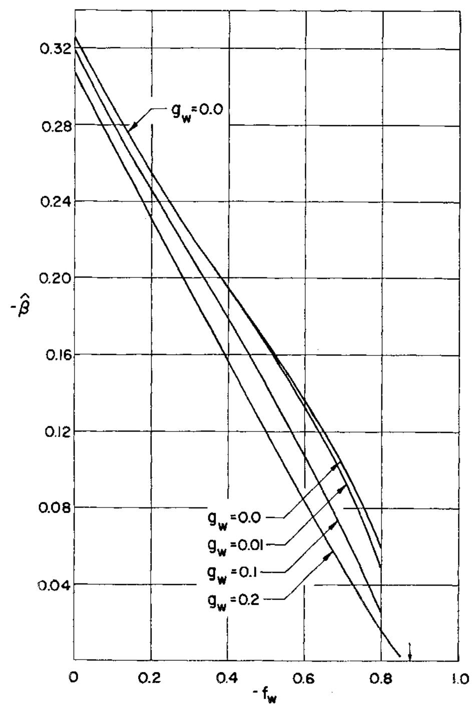

Fig. 1c Variation of pressure gradient parameter for $g_w \to 0$.

was carried out using the trapezoidal rule with a step-size, $\Delta\eta = 0.1$.‡ The maximum value of η is usually dependent on the blowing or suction rate. For all cases considered $\eta_{max} = 20$ was more than sufficient to obtain very smooth distributions for all the flow variables. For all cases, the initial profile was selected simply as

$$f'_{(0)} = 1$$

Even with this very crude selection, convergence was obtained very rapidly on the N.Y.U. CDC 6600. For cases where $f_w > 0$ and $-f_w$ small roughly 20–30 iterates were required to satisfy the error criteria of Eq. (23). As the Emmons-Leigh limiting value for f_w was approached we required nearly 500 iterates for $g_w \simeq 0$ and 100 iterates for $g_w \simeq 1.0$. Long running times then precluded extension of the map beyond $-f_w = 0.85$. Other numerical techniques could have been employed to facilitate convergence, e.g. quasilinearization, but for this particular problem were not required.

In a recent paper§ Kassoy[14] discussed the nature of the solution in the Emmons-Leigh limit. Indeed when $-f_w = 0.87574$, and since $\hat{\beta} = 0$ for $f''(0) = 0$, the wall boundary conditions imply vanishingly small derivatives of $f(\eta)$ everywhere. Numerical techniques based on a marching from the wall as in the approach presented here, then must necessarily fail. In Ref. 14 analytical tools, based on matched asymptotic expansions, are presented in detail to document this difficulty.

The selection of the above initial profile precludes identification of any of the lower branch solutions, noted in, among others, Ref. 3. Apparently, as pointed out by Libby and Liu,[15] these solutions, in general, cannot be obtained by standard numerical techniques. Related to this is the nonunique behavior with $\hat{\beta} < 0$; explicit suppression of an algebraically decaying term in the asymptotic solution of the momentum equation is required to obtain a unique solution. The assumed initial profile, in essence, suppresses this algebraic term; careful inspection of Eqs. (20–22) indicates that no such term can arise with the choice of $f'_{(0)}(\eta) = 1$.

‡ Several numerical tests were made, and based on accuracy requirements, this was deemed sufficient except for extremely large suction rates. This will be further discussed below.

§ The authors wish to thank an anonymous reviewer for calling their attention to this paper.

3. Results

In this section we investigate the results for the blowing and suction solutions separately.

A. Blowing, $f_w < 0$

The eigenvalues of the solution for $f_w < 0$ are presented in Table 1. For the selected values of blowing rate, f_w, and wall

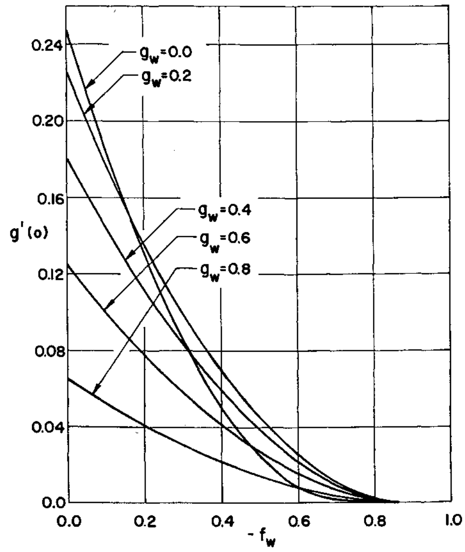

Fig. 2a Variation of heat-transfer parameter with blowing rate.

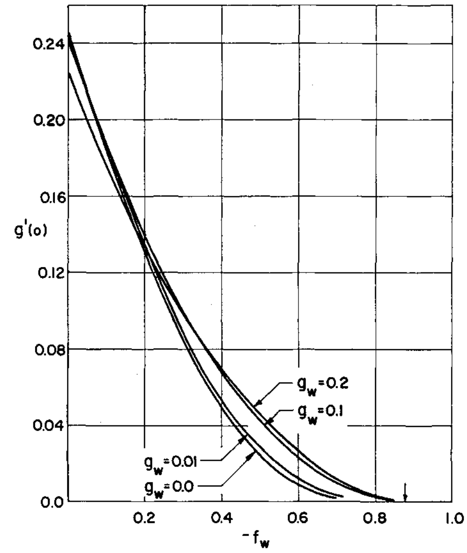

Fig. 2c Variation of heat-transfer parameter for $g_w \to 0$.

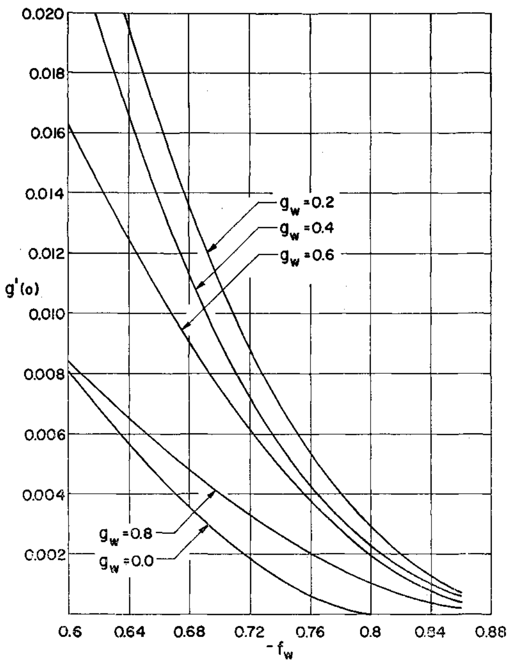

Fig. 2b Variation of heat-transfer parameter for $-f_w \to 0.87574$.

enthalpy ratio, g_w, the values obtained for $\hat{\beta}$, $g'(0)$, θ_{tr}, θ_{Htr}, δ^*_{tro}, and κ are displayed. The maps of the resulting pressure gradient parameter, $\hat{\beta}$, and heat-transfer rate, $g'(0)$ are indicated in Figs. 1a–c and 2a–c for various ranges of f_w and g_w.

Several observations are in order here. Consider first the solutions of the momentum equation as evidenced by the values of $\hat{\beta}$. Clearly, independent of the value of g_w, they all approach the Emmons and Leigh solution corresponding to $-f_w = 0.87574$ and $\hat{\beta} = 0$. However, the rate of approach thereto seems to be a strong function of the actual wall enthalpy ratio with the $g_w = 0$ results appearing somewhat pathological. The regular behavior of the solution can be observed by careful inspection of the results for $0 < g_w < 0.2$ displayed in Fig. 1c with smooth transition over the entire range of wall enthalpy ratios confirmed (at least numerically).

Similar remarks are applicable to the solutions of the energy equation, evidenced here by the values of $g'(0)$. Again the $g_w = 0$ case seems out of order but the numerical results displayed in Fig. 2c confirm a continuous behavior. Related commentary on other anomalous cold wall behavior may be found in Ref. 15.

Typical velocity and enthalpy profiles are indicated in Figs. 3–7. As to be expected when $-f_w \to 0.87574$ and $g_w \to 0$ the profiles thicken dramatically. Finally note that as the Emmons-Leigh limit is approached $\hat{\beta} \to 0$ and a Crocco integral exists in the form: $g = g_w + (1 - g_w)f'$; careful inspection of these figures indicates conformity with this.

B. Suction, $f_w < 0$

In Table 2 are displayed the corresponding eigenvalues for the suction solutions and in Figs. 8(a) and (b) and 9(a) and (b) the corresponding pressure gradient and heat-transfer maps. The typical profiles have also been indicated in Figs. 3–7.

Some interesting observations regarding the solutions for large f_w may be made in comparison to the classical asymptotic suction solution observed in Refs. 7 and 8 and the present minimum suction solutions, i.e., those with the minimum suction rate necessary to prevent separation. From the numerical solutions we may observe that the momentum equation appears relatively independent of the energy equation and that the solution for the wall heat-transfer rate is linearly dependent on f_w. A careful review of the pointwise output of the numerical solutions indicates that $f \simeq f_w$ over nearly the entire range of interest. This may also be inferred from the values of κ indicated in Table 2; for large, f_w, $\kappa \simeq -f_w$. We may use this notion to obtain a particularly simple solution to Eq. (2). Setting $f = f_w$ the equation is immediately integrable; satisfying the boundary conditions yields:

$$g = 1 - (1 - g_w) \exp(-f_w \eta) \tag{24a}$$

and

$$g'(0) = f_w(1 - g_w) \tag{24b}$$

In Figs. 9a and b the result of Eq. (24b) is indicated as the asymptotic solution. Extremely good agreement is obtained even when $f_w \simeq 1$. For very large f_w the preceding solution appears to diverge from the indicated numerical result; we attribute this to some inaccuracies in the step size selected for integration. Note that for $f_w \approx 10$ the asymptotic solution is obtained near $\eta \simeq 2$ and consequently the requirement of a fine integration mesh is indicated. Equation (24) is identical to the actual asymptotic suction solution presented by Morduchow[8] for the energy equation. Apparently the drive toward this form of the solution is very strong when the suction rates become high, independent of the nature of the velocity field.

Table 2 Eigenvalues for suction solutions

g_w	f_w	$-\hat{\beta}$	$g'(0)$	δ_{tro}^*	θ_{tr}	θ_{Htr}	κ
0.0	0.2	0.40577	0.38800	1.3737	0.60143	0.18800	2.8526
	0.4	0.49477	0.54525	1.3883	0.56779	0.14525	2.3477
	0.6	0.59390	0.71437	1.3779	0.53760	0.11437	1.9064
	0.8	0.70317	0.89163	1.3528	0.50948	0.09163	1.5094
	1.0	0.82246	1.0745	1.3195	0.48015	0.07449	1.1449
	1.4	1.0908	1.4506	1.2428	0.48914	0.05062	0.48393
	1.8	1.3978	1.8344	1.1646	0.43251	0.03437	−0.1152
	2.0	1.5657	2.0277	1.1270	0.41629	0.02770	−0.39847
	4.0	3.7726	4.0001	0.83932	0.30064	0.00006	−2.9129
	6.0	6.8948	5.9631	0.66222	0.24329	−0.03688	−5.1706
	8.0	10.928	7.9004	0.54458	0.20638	−0.09964	−7.3290
	10.0	15.848	9.8008	0.46132	0.18112	−0.19917	−9.4367
0.2	0.2	0.39248	0.33386	1.4283	0.59351	0.17386	2.4678
	0.4	0.48435	0.45558	1.4242	0.56199	0.13558	2.0519
	0.6	0.58534	0.58745	1.4029	0.53333	0.10745	1.6695
	0.8	0.69587	0.72649	1.3711	0.50661	0.8649	1.3135
	1.0	0.81614	0.87060	1.3333	0.47956	0.07060	0.97866
	1.4	1.0856	1.1684	1.2517	0.48014	0.04840	0.35754
	1.8	1.3935	1.4734	1.1710	0.42747	0.03341	−0.21660
	2.0	1.5616	1.6274	1.1326	0.4118	0.02735	−0.49050
	4.0	3.7702	3.2021	0.84118	0.29916	0.00212	−2.9609
	6.0	6.8930	4.7717	0.66368	0.24186	−0.02833	−5.2027
	8.0	10.926	6.3211	0.54610	0.20485	−0.07893	−7.3528
	10.0	15.846	7.8412	0.46299	0.17943	−0.15876	−9.4554
0.4	0.2	0.36532	0.26149	1.5562	0.58064	0.14149	2.2426
	0.4	0.46051	0.35191	1.5142	0.55104	0.11191	1.8563
	0.6	0.56432	0.44969	1.4680	0.52422	0.08969	1.4988
	0.8	0.67708	0.55283	1.4197	0.49931	0.07283	1.1635
	1.0	0.79914	0.65988	1.3707	0.47484	0.05988	0.84575
	1.4	1.0714	0.88157	1.2753	0.46993	0.04157	0.25014
	1.8	1.3812	1.1091	1.1873	0.41985	0.02914	−0.30586
	2.0	1.5502	1.2241	1.1465	0.40484	0.02413	−0.57273
	4.0	3.7631	2.4031	0.84538	0.29633	0.00311	−3.0064
	6.0	6.8872	3.5796	0.66636	0.23961	−0.02038	−5.2335
	8.0	10.921	4.7414	0.54835	0.20274	−0.05861	−7.3759
	10.0	15.841	5.8814	0.46517	0.17730	−0.11864	−9.4737
0.6	0.2	0.33501	0.17927	1.7262	0.56887	0.09927	2.0993
	0.4	0.43119	0.23951	1.6399	0.53992	0.07951	1.7218
	0.6	0.53640	0.30442	1.5628	0.51403	0.06442	1.3742
	0.8	0.65077	0.37281	1.4925	0.49044	0.05281	1.0485
	1.0	0.77433	0.44376	1.4278	0.46805	0.04376	0.7397
	1.4	1.0404	0.59079	1.3124	0.45973	0.03079	0.1597
	1.8	1.3616	0.74189	1.2130	0.41039	0.02189	−0.38390
	2.0	1.5315	0.81831	1.1684	0.39618	0.01831	−0.64557
	4.0	3.7510	1.6031	0.85198	0.29235	0.00306	−3.0494
	6.0	6.8783	2.3870	0.67013	0.23657	−0.01301	−5.2631
	8.0	10.914	3.1613	0.55126	0.20005	−0.03869	−7.3982
	10.0	15.834	3.9212	0.46785	0.17476	−0.07881	−9.4914
0.8	0.2	0.30602	0.09137	1.9218	0.55924	0.05137	2.0024
	0.4	0.40092	0.12156	1.7898	0.53011	0.04156	1.6263
	0.6	0.50574	0.15399	1.6794	0.50445	0.03399	1.2817
	0.8	0.62030	0.18810	1.5844	0.48148	0.02810	0.95992
	1.0	0.74442	0.22346	1.5014	0.46043	0.02346	0.65548
	1.4	1.0212	0.29673	1.3615	0.45268	0.01673	0.08409
	1.8	1.3353	0.37206	1.2476	0.39984	0.01206	−0.45151
	2.0	1.5062	0.41017	1.1980	0.38643	0.01017	−0.70946
	4.0	3.7333	0.80201	0.86101	0.28743	0.00201	−3.0897
	6.0	6.8652	1.1938	0.67504	0.23285	−0.00622	−5.2916
	8.0	10.903	1.5808	0.55490	0.19688	−0.01915	−7.4198
	10.0	15.824	1.9607	0.47101	0.17182	−0.03926	−9.5086
1.0	0.2	0.27991	0.0	2.1335	0.55155	0.0	1.9334
	0.4	0.37203	0.0	1.9562	0.52189	0.0	1.5561
	0.6	0.47494	0.0	1.8117	0.49602	0.0	1.2116
	0.8	0.58832	0.0	1.6910	0.47320	0.0	0.89094
	1.0	0.71181	0.0	1.5882	0.45282	0.0	0.58819
	1.4	0.98846	0.0	1.4212	0.42029	0.0	0.02125
	1.8	1.3035	0.0	1.2903	0.38870	0.0	−0.50966
	2.0	1.4750	0.0	1.2348	0.37610	0.0	−0.76523
	4.0	3.7095	0.0	0.87250	0.28177	0.0	−3.1275
	6.0	6.8468	0.0	0.68115	0.22854	0.0	−5.3188
	8.0	10.888	0.0	0.55923	0.19325	0.0	−7.4407
	10.0	15.827	0.0	0.47417	0.16830	0.0	−9.5258
	20.0	55.638	0.0	0.26779	0.09335	0.0	−19.732

Table 2 (*continued*)

g_w	f_w	$-\hat{\beta}$	$g'(0)$	δ_{tro}^*	θ_{tr}	θ_{Htr}	κ
1.5	0.2	0.22783	−0.23664	2.7022	0.53829	−0.13664	1.8270
	0.4	0.31047	−0.31290	2.4144	0.50703	−0.11290	1.4445
	0.6	0.40512	−0.39432	2.1859	0.48002	−0.09432	1.0969
	0.8	0.51145	−0.47961	2.0003	0.45655	−0.07961	0.77476
	1.0	0.62922	−0.56780	1.8464	0.43631	−0.06780	0.47156
	1.4	0.89774	−0.75018	1.6064	0.41144	−0.05018	−0.09355
	1.8	1.2085	−0.93756	1.4276	0.35906	−0.03756	−0.62051
	2.0	1.3789	−1.0324	1.3543	0.34981	−0.0326	−0.87365
	4.0	3.6206	−2.0088	0.91246	0.26575	−0.00881	−3.2099
	6.0	6.7768	−2.9868	0.70154	0.21567	0.1320	−5.3816
	8.0	10.831	−3.9538	0.57305	0.18240	0.04624	−7.4899
	10.0	15.760	−4.9031	0.48573	0.15886	0.09694	−9.5651
2.0	0.2	0.19063	−0.47911	3.2998	0.53013	−0.27911	1.7671
	0.4	0.26374	−0.63245	2.9056	0.49752	−0.23245	1.3804
	0.6	0.36891	−0.79582	2.5955	0.46935	−0.1958	1.0296
	0.8	0.44609	−0.96669	2.3459	0.44499	−0.16669	0.70492
	1.0	0.55512	−1.14320	2.1413	0.42410	−0.14320	0.39989
	1.4	0.80796	−1.50783	1.8269	0.39604	−0.10783	−0.16737
	1.8	1.1058	−1.8822	1.59737	0.31707	−0.08218	−0.69496
	2.0	1.2710	−2.07155	1.50465	0.32291	−0.07155	−0.94798
	4.0	3.4920	−4.0222	0.96781	0.24897	−0.02218	−3.2763
	6.0	6.6616	−5.9767	0.72985	0.20101	0.02330	−5.4362
	8.0	10.734	−7.9098	0.59142	0.16969	0.09024	−7.5345
	10.0	15.679	−9.8078	0.49971	0.14751	0.19219	−9.7020

It is of interest to obtain a similar asymptotic solution for the momentum equation. Unfortunately, even recognizing the independence of the energy equation, i.e., taking $g \equiv 1$, the nonlinearity present in the term f'^2 precludes a simple analysis. Some indication of the form of the relation $\hat{\beta} = \hat{\beta}(f_w)$ can be obtained from an integral method analysis of Prandtl. This solution is reported in detail in Ref. 7 and in terms of the present notation may be written simply as:

$$f_w \to \infty, \quad -\hat{\beta} = 0.21 f_w^2 \tag{25}$$

Within the present context additional information can be obtained from expansion of the momentum equation for small and large values of η. In the following we employ the numerical result that solutions to Eq. (1) are relatively insensitive to values of g_w and attempt to derive a solution for $g \equiv 1$. In the region near $\eta \simeq 0$ we employ the same approximation used to derive Eqs. 24, namely we set $f \simeq f_w =$ constant. In this region we may neglect f'^2 compared to 1 and the differential equation Eq. (1) becomes:

$$f''' + f_w f'' = -\hat{\beta} \tag{26}$$

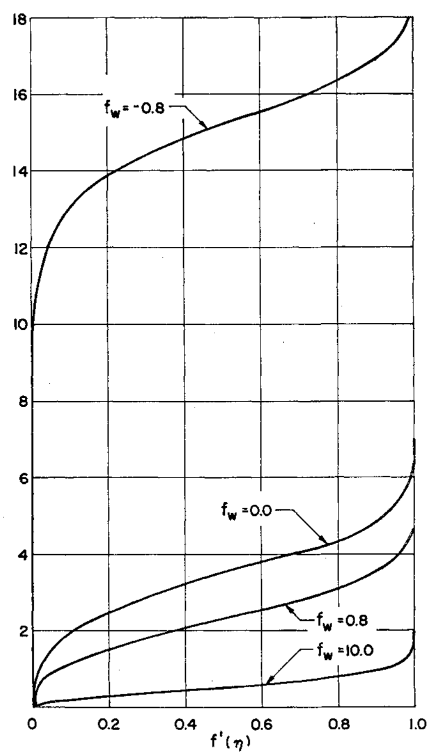

Fig. 3 Typical velocity profiles, $g_w = 0$.

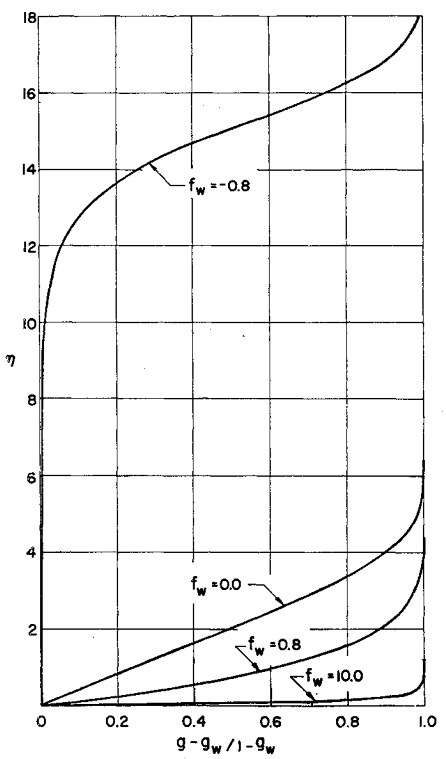

Fig. 4 Typical energy profiles, $g_w = 0$.

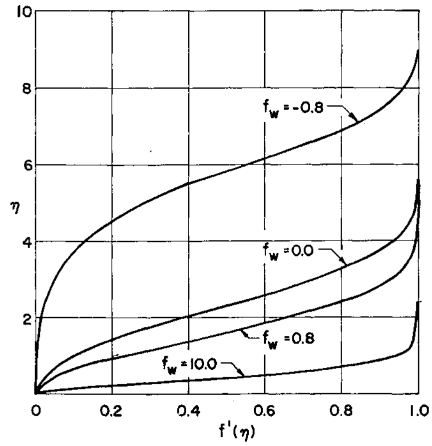

Fig. 5 Typical velocity profiles, g_w = 1.0.

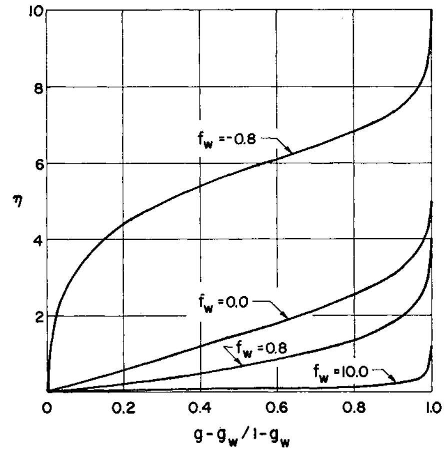

Fig. 7 Typical energy profiles, g_w = 2.0.

The solution thereof is

$$f'' = (-\hat{\beta}/f_w)[1 - \exp(-f_w\eta)] \tag{27a}$$

$$f' = (-\hat{\beta}/f_w^2)[f_w\eta - 1 + \exp(-f_w\eta)] \tag{27b}$$

Investigation of the range of validity of these relations indicates that they may be used with good accuracy for values of $f_w\eta \lesssim 3$.

The corresponding solution for η large may be found in Ref. 3. Again for $g \equiv 1$ we have¶:

$$f'' = -[(2\hat{\beta} + 1)(\eta - \kappa)^{-2} + 1]\alpha_1(\eta - \kappa)^{-2\beta} \times \exp[-(\eta - \kappa)^2/2] \tag{28a}$$

$$f' = 1 + (\eta - \kappa)^{-1}\alpha_1(\eta - \kappa)^{-2\beta}\exp[-(\eta - \kappa)^2/2] \tag{28b}$$

Simple patching of these two solutions appears to yield a result similar to that suggested by Prandtl, i.e.,

$$-\hat{\beta} \simeq af_w^2(1 + b/f_w) \tag{29}$$

Unfortunately, the values of a and b are largely affected by the actual value of η at the patch. Rather than investigating this further we can curve-fit the data and determine that $a = 0.11$, $b = 4.0$. The results indicated by Eq. (25) and (29) are displayed in Fig. 8b where good agreement is obtained with this latter approximation.

In a recent paper received after this work was completed, Kassoy and Libby[9] have investigated the nature of this solution using the method of matched asymptotic expansions.

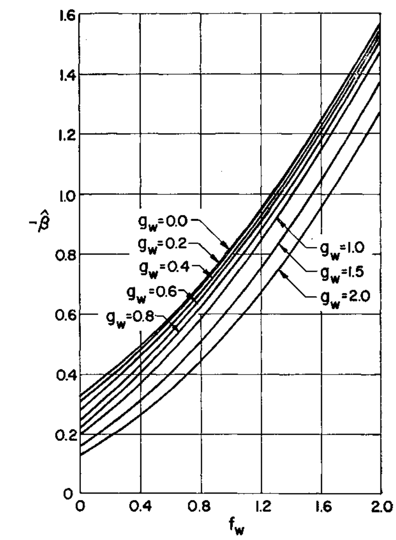

Fig. 8a Variation of pressure gradient parameter with suction rate.

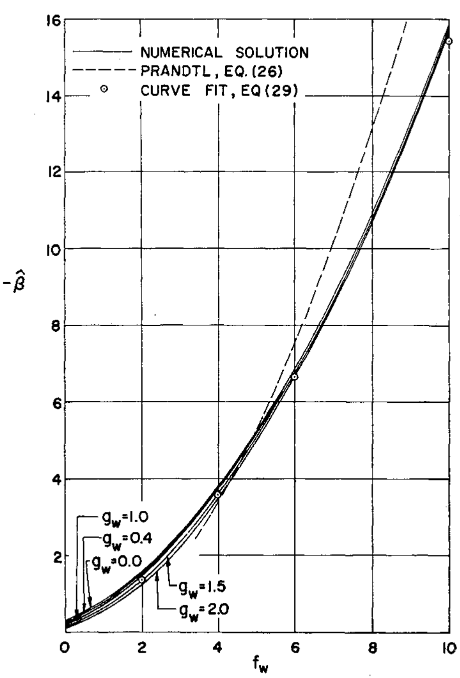

Fig. 8b Variation of pressure gradient parameter for f_w large.

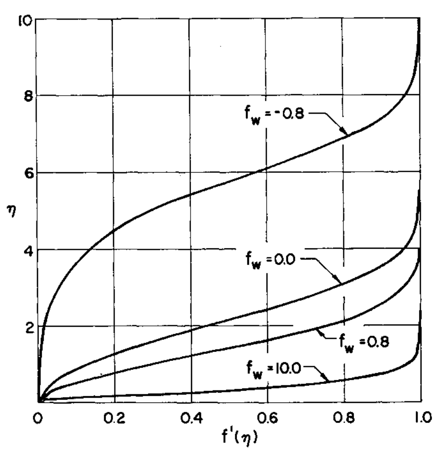

Fig. 6 Typical velocity profiles, g_w = 2.0.

in¶ Note that we have suppressed the algebraic decay indicated Ref. 3.

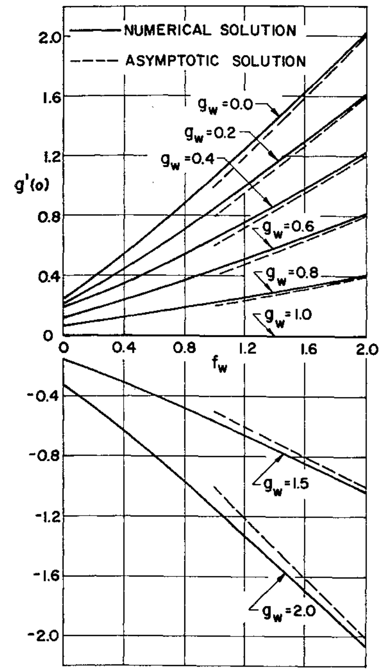

Fig. 9a Variation of heat-transfer parameter with suction rate.

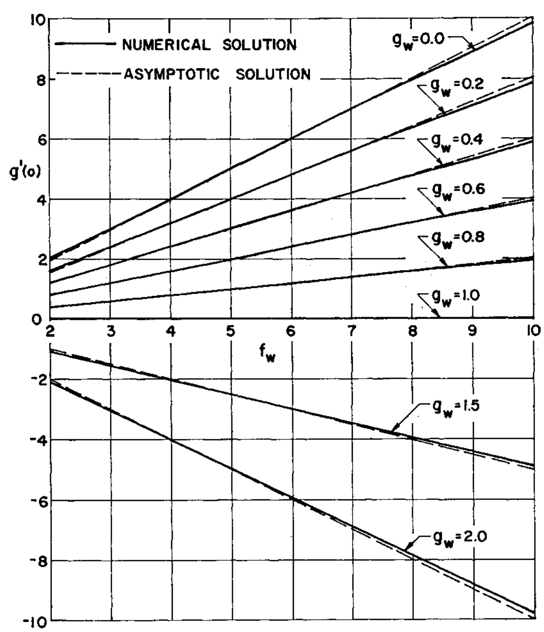

Fig. 9b Variation of heat-transfer parameter for f_w large.

They find that

$$-\hat{\beta} \simeq f_w^2/8 - k_1$$

where k_1 is a constant. While functionally quite different from Eq. 29 the leading terms are remarkably close and their agreement with the present results is excellent.

4. Concluding Remarks

A complete map of separation solutions to the Cohen-Reshotko equations have been obtained. This map takes the form of the pressure gradient parameter, $\hat{\beta}$ and heat transfer parameter, $g'(0)$, as functions of the mass transfer rate and wall enthalpy ratio.

Solutions up to the Emmons-Leigh value for the blowing rate have been obtained. In the suction regime an asymptotic solution has been recognized and solutions thereof obtained.

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